EMTP II   2. Klausur   02.03.2011   M.L.

1.

(a) $\vec{B} = (\nabla u) \times (\nabla v) = \vec{e}_k \, \varepsilon_{ijk} \, (\partial_i u)(\partial_j v)$

$$\nabla \cdot \vec{B} = \partial_k B_k = \varepsilon_{ijk} \partial_k \left\{ (\partial_i u)(\partial_j v) \right\} = \underbrace{\varepsilon_{ijk}}_{\text{antisymm.}} \left\{ (\underbrace{\partial_k \partial_i u}_{\text{symm.}})(\partial_j v) + (\partial_i u)(\underbrace{\partial_k \partial_j v}_{\text{symm.}}) \right\} = 0 \qquad \square .$$

(3 Punkte)

(b) $\vec{A} = \frac{1}{2} \left(u \nabla v - v \nabla u \right) = \vec{e}_k \cdot \frac{1}{2} \left\{ u \partial_k v - v \partial_k u \right\}$

$$\nabla \times \vec{A} = \vec{e}_k \, \varepsilon_{kij} \, \partial_i A_j = \vec{e}_k \, \varepsilon_{kij} \, \frac{1}{2} \, \partial_i \left\{ u \partial_j v - v \partial_j u \right\}$$

$$= \vec{e}_k \, \varepsilon_{kij} \cdot \frac{1}{2} \left\{ \partial_i u \partial_j v + \underbrace{u \partial_i \partial_j v}_{\text{symm.}} - \underbrace{\partial_i v \partial_j u}_{\text{umnenne Indizes}} - \underbrace{v \partial_i \partial_j u}_{\text{symm.}} \right\}$$

$$= \vec{e}_k \, \varepsilon_{kij} \frac{1}{2} \left\{ \partial_i u \partial_j v + \partial_j v \partial_i u \right\}$$

$$= \vec{e}_k \, \varepsilon_{kij} \, \partial_i u \partial_j v \; . \qquad \square .$$

(3 Punkte)

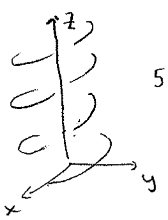
2.

$x = \cos\varphi$

$y = \sin\varphi$

$z = \frac{\varphi}{2\pi}$

$0 \le \varphi \le 10\pi$



5 volle Runden

(1 Punkt)

(a) $$\vec{e}_z \cdot \int_C d\vec{r} \times \vec{E} = \int_0^{10\pi} d\varphi \, \frac{d\vec{r}}{d\varphi} \times \vec{r}$$

$\vec{r} = \left(\cos\varphi, \sin\varphi, \frac{\varphi}{2\pi} \right)$

$\frac{d\vec{r}}{d\varphi} = \left(-\sin\varphi, \cos\varphi, \frac{1}{2\pi} \right)$

(1 Punkt)

$$\frac{d\vec{r}}{d\varphi} \times \vec{r} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -\sin\varphi & \cos\varphi & \frac{1}{2\pi} \\ \cos\varphi & \sin\varphi & \frac{\varphi}{2\pi} \end{vmatrix} = \vec{e}_x \left(\frac{\varphi}{2\pi} \cos\varphi - \frac{1}{2\pi} \sin\varphi \right) + \vec{e}_y \left(\frac{1}{2\pi} \cos\varphi + \frac{\varphi}{2\pi} \sin\varphi \right) + \vec{e}_z \left(-\sin^2\varphi - \cos^2\varphi \right)$$

$$\vec{e}_z \cdot \frac{d\vec{r}}{d\varphi} \times \vec{r} = -1$$

(1 Punkt)

$$\Rightarrow \vec{e}_z \cdot \int d\vec{r} \times \vec{E} = -\int_0^{10\pi} d\varphi = -10\pi$$

(1 Punkt)

(b) $$\int_C d\vec{r} \cdot \vec{B} = \int_0^{10\pi} d\varphi \, \frac{d\vec{r}}{d\varphi} \cdot \vec{B}$$

$\frac{d\vec{r}}{d\varphi} = \left(-\sin\varphi, \cos\varphi, \frac{1}{2\pi} \right)$

$\vec{B} = \left(\sin\varphi, \cos\varphi, \frac{\varphi}{2\pi} \right)$

$$\frac{d\vec{r}}{d\varphi} \cdot \vec{B} = -\sin^2\varphi + \cos^2\varphi + \frac{\varphi}{4\pi^2} = \cos 2\varphi + \frac{\varphi}{4\pi^2}$$

(1 Punkt)

$$\int_0^{10\pi} d\varphi \left\{ \cos 2\varphi + \frac{\varphi}{4\pi^2} \right\} = \left[\overset{0}{\cancel{\frac{1}{2} \sin 2\varphi}} + \frac{\varphi^2}{8\pi^2} \right]_0^{10\pi} = \frac{100}{8} = \frac{25}{2}$$

(1 Punkt)

③ $\vec{E} = \dfrac{\vec{r}\,\sqrt{x^2+y^2}}{r^4}$

(a) z.B. direkt in kartesischen Koordinaten:

$$E_x = \frac{x\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} \; ; \quad \partial_x E_x = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} + \frac{x^2}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^2} - 4\,\frac{x^2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^3}$$

$$E_y = \frac{y\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} \; ; \quad \partial_y E_y = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} + \frac{y^2}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^2} - 4\,\frac{y^2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^3}$$

$$E_z = \frac{z\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} \; ; \quad \partial_z E_z = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} - 4\,\frac{z^2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^3}$$

$$\frac{3\sqrt{x^2+y^2}}{r^4} + \frac{\sqrt{x^2+y^2}}{r^4} - 4\,\frac{\sqrt{x^2+y^2}}{r^4} = 0 \quad \square$$

(3 Punkte)

(b) $\nabla\cdot\vec{E} = 0$ für $\vec{r} \neq \vec{0}$

Ursprung innerhalb von S und S', so dass keine Singularitäten zwischen S, S'

(1 Punkt)

$$I' = \oint_{S'} d\vec{A}\cdot\vec{E}$$

$$I = \oint_S d\vec{A}\cdot\vec{E}$$

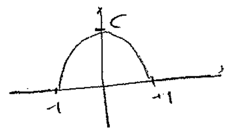
$$I' - I = \oint_{S'\cup S} d\vec{A}\cdot\vec{E} \overset{\text{Gauß}}{=} \int dV\, \nabla\cdot\vec{E} = 0 \quad \square$$

Oberflächenelement „nach innen"

Volumen zwischen S', S

(2 Punkte)

④ $f(x) = \begin{cases} c\,(1-x^2), & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$



(a)
$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} = c\int_{-1}^{+1} dx\,(1-x^2)\, e^{-ikx}$$

$$= c\int_{-1}^{1} dx \left\{ -\frac{1}{ik}\frac{d}{dx}e^{-ikx} + \frac{1}{ik}\frac{d}{dx}\left(x^2 e^{-ikx}\right) - \frac{2}{ik}\, x\, e^{-ikx} \right\}$$

$$= c\int_{-1}^{+1} dx \left\{ -\frac{1}{ik}\frac{d}{dx}e^{-ikx} + \frac{1}{ik}\frac{d}{dx}\left(x^2 e^{-ikx}\right) - \frac{2}{k^2}\frac{d}{dx}\left(x e^{-ikx}\right) + \frac{2}{k^2} e^{-ikx} \right\}$$

$$= c\left[\underbrace{-\frac{1}{ik}e^{-ikx} + \frac{1}{ik}x^2 e^{-ikx}}_{\text{heben auf}} - \frac{2}{k^2}\, x\, e^{-ikx} - \frac{2}{ik^3}\, e^{-ikx} \right]_{-1}^{+1}$$

$$= c\left[-\frac{2}{k^2}\left(e^{-ik} + e^{ik}\right) - \frac{2}{ik^3}\left(e^{-ik} - e^{ik}\right) \right]$$

$$= -\frac{4c}{k^2}\left[\cos k - \frac{\sin k}{k}\right] = 4c\left[\frac{\sin k}{k^3} - \frac{\cos k}{k^2}\right]$$

(4 Punkte)

(b)
$$\delta_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) = \begin{cases} c\left(1 - \frac{x^2}{\varepsilon^2}\right), & |x|<\varepsilon \\ 0, & |x|\geq\varepsilon \end{cases}$$

Soll gelten: $1 \doteq \int_{-\infty}^{\infty} dx\, \delta_\varepsilon(x) = \int_{-\infty}^{\infty} dx\, f\left(\frac{x}{\varepsilon}\right) = \varepsilon \int_{-\infty}^{\infty} dx\, f(x) = \varepsilon \cdot \lim_{k\to 0} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$

D.h. $1 = \varepsilon\cdot 4c\left[\dfrac{k - \frac{k^3}{6}}{k^3} - \dfrac{1 - \frac{k^2}{2}}{k^2}\right] = \varepsilon\cdot 4c\cdot\left(\frac{1}{2} - \frac{1}{6}\right) = \varepsilon\cdot 4c\cdot\frac{1}{3}$

$$\boxed{c = \frac{3}{4\varepsilon}}$$

(2 Punkte)