EMTP II / 1. Klausur / 02.09.2010 / M.L.

1.

(a) $\vec{E} = (2axy + z^3)\,\vec{e}_x + x^2\,\vec{e}_y + 3axz^2\,\vec{e}_z$

[2 Punkte]

$$\nabla\times\vec{E} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ 2axy+z^3 & x^2 & 3axz^2 \end{vmatrix} = \vec{e}_x\cdot 0 + \vec{e}_y\left(3z^2 - 3az^2\right) + \vec{e}_z\left(2x - 2ax\right) \overset{!}{=} 0$$

für $\underline{\underline{a = 1}}$.

(b)
$$\Gamma = \oint d\vec{r}\cdot\vec{E} = \int_0^{2\pi} d\phi\, \frac{d\vec{r}}{d\phi}\cdot\vec{E} \qquad \frac{d\vec{r}}{d\phi} = \begin{pmatrix} -3\sin\phi \\ -4\sin\phi \\ 5\cos\phi \end{pmatrix} \qquad \vec{E} = \begin{pmatrix} 24a\,(1+\cos\phi)^2 + 125\sin^3\phi \\ 9\,(1+\cos\phi)^2 \\ 225a\,(1+\cos\phi)\sin^2\phi \end{pmatrix}$$

$$\Rightarrow \Gamma = \int_0^{2\pi} d\phi \left\{ \begin{matrix} -72a\,(1+\cos\phi)^2\sin\phi & -375\sin^4\phi \\ -36\,(1+\cos\phi)^2\sin\phi & +1125a\,(1+\cos\phi)\cos\phi\sin^2\phi \end{matrix} \right\}$$

$$= \int_0^{2\pi} d\phi \left\{ -36(1+2a)\big(\underbrace{\cancel{\sin\phi}}_{0} + \underbrace{2\cancel{\cos\phi}\sin\phi}_{0} + \underbrace{\cos^2\phi\cancel{\sin\phi}}_{0}\big) - 375\sin^2\phi\,(1-\cos^2\phi) + 1125a\,\big(\underbrace{\cancel{\cos\phi}\sin^2\phi}_{0} + \cos^2\phi\sin^2\phi\big) \right\}$$

$$\int_0^{2\pi} d\phi\,\sin^2\phi = \frac{1}{2}\int_0^{2\pi} d\phi\,(\sin^2\phi + \cos^2\phi) = \pi$$

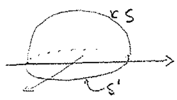
$$\int_0^{2\pi} d\phi\,\sin^2\phi\cos^2\phi = \frac{1}{4}\int_0^{2\pi}\sin^2 2\phi = \frac{1}{8}\int_0^{2\pi}\left(\sin^2 2\phi + \cos^2 2\phi\right) = \frac{\pi}{4}$$

[3 Punkte]
$$\Rightarrow \Gamma = \pi\left\{-375 + \frac{375}{4} + \frac{1125a}{4}\right\} = \frac{\pi}{4}\left\{-1125 + 1125a\right\} = \underline{\underline{\frac{1125\pi}{4}(a-1)}}$$

[1 Punkt] Verschwindet für $a = 1$.

2.

$\vec{B} = B\vec{e}_z$



$$\oint_{\partial V} d\vec{A}\cdot\vec{B} \underset{\substack{\uparrow \\ \text{Gauß}}}{=} \int_V dV\,\nabla\cdot\vec{B} = 0 \quad\Rightarrow\quad \int_S d\vec{A}\cdot\vec{B} = \int_{S'} d\vec{A}\cdot\vec{B} = B\cdot\pi R^2$$

Die Aufgabe war aber, $\int_S d\vec{A}\cdot\vec{B}$ explizit zu berechnen!

Kugelkoordinaten:

[2 Punkte]

$$\vec{r} = R\,(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$
$$\partial_\theta\vec{r} = R\,(\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)$$
$$\partial_\varphi\vec{r} = R\,(-\sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ 0)$$
$$\partial_\theta\vec{r}\times\partial_\varphi\vec{r} = R^2\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ c\theta c\varphi & c\theta s\varphi & -s\theta \\ -s\theta s\varphi & s\theta c\varphi & 0 \end{vmatrix} = \ldots + R^2\vec{e}_z\left(\cos\theta\sin\theta\cos^2\varphi + \cos\theta\sin\theta\sin^2\varphi\right)$$
$$= \ldots + R^2\vec{e}_z\,\cos\theta\sin\theta$$

$$\underbrace{\int_S d\vec{A}\cdot\vec{B} = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\varphi\, \underbrace{R^2\cos\theta\sin\theta\,B}_{\partial_\theta\vec{r}\times\partial_\varphi\vec{r}\,\cdot\,\vec{B}}}_{[2\text{ Punkte}]} = BR^2\cdot 2\pi\cdot\int_0^{\frac{\pi}{2}} d\theta\,\sin\theta\cos\theta$$

$$= BR^2\cdot\pi\cdot\left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = \underbrace{\underline{\underline{B\cdot\pi R^2}}}_{[2\text{ Punkte}]}$$

3.

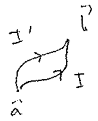
(a) $\vec{F} = \frac{1}{(x^2+y^2)^{3/2}} (x\,\vec{e}_x + y\,\vec{e}_y)$

[2 Punkte]

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{\rho^3} & \frac{y}{\rho^3} & 0 \end{vmatrix} = \vec{e}_x \underbrace{\left(-\partial_z \frac{y}{\rho^3}\right)}_{0} + \vec{e}_z \underbrace{\left(\partial_x \frac{y}{\rho^3} - \partial_y \frac{x}{\rho^3}\right)}_{-3y\cdot\frac{x}{\rho^5} + 3x\cdot\frac{y}{\rho^5} = 0} + \vec{e}_y \underbrace{\left(\partial_z \frac{x}{\rho^3}\right)}_{0} \qquad \square$$

(b)

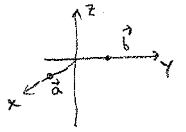
[2 Punkte]



$$I - I' = \oint d\vec{r}\cdot\vec{F} \underset{\text{Stokes}}{=} \int_S d\vec{A}\cdot \underbrace{\nabla\times\vec{F}}_{0} = 0 \qquad \square$$

(c)

[2 Punkte]



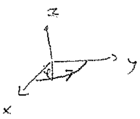
Wähle Weg durch Ursprung

$$\int_1^0 dx\, \vec{e}_x \cdot \vec{F}\big|_{z=y=0} + \int_0^1 dy\, \vec{e}_y \cdot \vec{F}\big|_{x=z=0}$$

$$= -\int_0^1 dx \frac{x}{x^3} + \int_0^1 dy \frac{y}{y^3} \overset{?}{=} 0$$

Aber divergente Integrale, d.h. nicht definiert!

Besser:



$\varphi \in (0, \frac{\pi}{2})$

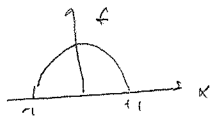
$\vec{r} = \underbrace{\cos\varphi}_{x}\, \vec{e}_x + \underbrace{\sin\varphi}_{y}\, \vec{e}_y \quad ; \quad \frac{d\vec{r}}{d\varphi} = -\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y$

$$I = \int_0^{\frac{\pi}{2}} d\varphi \frac{d\vec{r}}{d\varphi} \cdot \underbrace{(\cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y)}_{\vec{F}} = \int_0^{\pi/2} d\varphi\, (-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi) = 0.$$

Am besten: $\vec{F} = -\nabla (x^2+y^2)^{-1/2} \Rightarrow \int_{\vec{a}}^{\vec{b}} d\vec{r}\cdot\vec{E} = -(x^2+y^2)^{-1/2}\big|_{\vec{b}} + (x^2+y^2)^{-1/2}\big|_{\vec{a}} = 0.$

4.

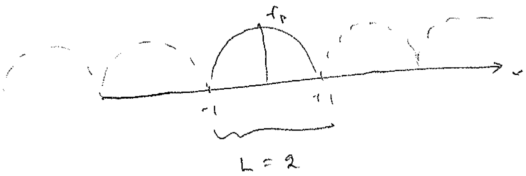
$f(x) = 1 - x^2, \quad |x| < 1$



(a)

[4 Punkte]

3: im Prinzip richtig
1: richtige Antwort



$$c_n = \frac{1}{2} \int_{-1}^{+1} dx\, (1-x^2)\, e^{-i\pi n x}$$

$$n = 0 \Rightarrow c_0 = \frac{1}{2}\int_{-1}^{+1} dx\,(1-x^2) = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{+1} = \frac{2}{3}$$

$n \neq 0$

$$\int_{-1}^{+1} dx\, e^{-i\pi n x} = \frac{1}{-i\pi n}\left[e^{-i\pi n x}\right]_{-1}^{+1} = \frac{1}{-i\pi n}\left[(-1)^n - (-1)^n\right] = 0$$

$$\int_{-1}^{+1} dx\, x^2 e^{-i\pi n x} = \underbrace{\frac{1}{-i\pi n}\left[x^2 e^{-i\pi n x}\right]_{-1}^{+1}}_{0} + \frac{2}{i\pi n}\int_{-1}^{+1} dx\, x\, e^{-i\pi n x}$$

$$= \frac{2}{\pi^2 n^2}\left[x e^{-i\pi n x}\right]_{-1}^{+1} - \frac{2}{\pi^2 n^2}\underbrace{\int_{-1}^{+1} dx\, e^{-i\pi n x}}_{0}$$

$$= \frac{4}{\pi^2 n^2}(-1)^n$$

$$\leadsto \underline{\underline{1 - x^2}} = \frac{2}{3} - \frac{1}{2}\sum_{n\neq 0} \frac{4}{\pi^2 n^2}(-1)^n e^{i\pi n x} = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(-1)^n \cos(\pi n x)$$

(b) Wähle $x = 0 \Rightarrow 1 = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

[2 Punkte]

$1\frac{1}{2}$: im Prinzip richtig
$\frac{1}{2}$: richtige Antwort

$$\underline{\underline{-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}}$$