Probeklausur 01.02.2010

1.

$$y^3 + x^3 + xy^2 + yx^2 = 0 \qquad (*)$$

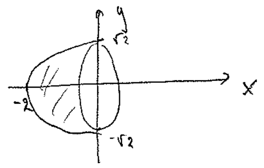
$y(1) = ?$ Setze $x = 1$ $\Rightarrow y^3 + y^2 + y + 1 = 0 \Leftrightarrow (y+1)(y^2+1) = 0 \Rightarrow \underline{\underline{y = -1}}$.

$y'(1) = ?$ Ableite $(*)$ $\Rightarrow$ $3y^2y' + 3x^2 + y^2 + 2xyy' + y'x^2 + 2xy = 0$

$$y'(3y^2 + 2xy + x^2) = -3x^2 - y^2 - 2xy$$

$$y' = -\frac{3x^2 + y^2 + 2xy}{3y^2 + x^2 + 2xy} = -\frac{3+1-2}{3+1-2} = \underline{\underline{-1}} \ .$$

2.



$$x = y^2 - 2$$

$$y^2 = x + 2 \qquad y = \sqrt{x+2} =: f(x)$$

$$V = \int_{-2}^{0} dx\, \pi [f(x)]^2 = \int_{-2}^{0} dx\, \pi\, (x+2) = \pi \left[\frac{x^2}{2} + 2x\right]_{-2}^{0}$$

wie in Aufgabe 6.2

$$= \pi \left\{ -\frac{(-2)^2}{2} + 4 \right\} = \underline{\underline{2\pi}}$$

3.

$$y' + y\cos x = \cos x$$

homogene Gleichung:

$$\frac{dy}{dx} = -y \cos x$$

$$\frac{dy}{y} = -\cos x\, dx \qquad d \ln|y| = -d \sin x$$

$$\ln|y| = -\sin x + c_0$$

$$|y| = \exp(-\sin x) \exp(c_0)$$

$$\Rightarrow \underline{\underline{y_a = C \exp(-\sin x)}}$$

Variation der Konstanten:

$$y(x) = C(x) \exp(-\sin x)$$

$$y'(x) = C'(x) \exp(-\sin x) - \cos x\, C(x) \exp(-\sin x)$$

$$\Rightarrow y'(x) + y(x)\cos x = C'(x) \exp(-\sin x)$$

$$C'(x) = \exp(+\sin x) \cos x$$

$$C(x) = \int^{x} dy\, \underbrace{\exp(+\sin y) \cos y}_{\frac{d}{dy}\exp(\sin y)} = \left[\exp(\sin(y))\right]^{x} = \exp(\sin(x)) - \text{Const.}$$

$$\Rightarrow y_s = C(x) \exp(-\sin(x)) = 1 - \underbrace{\text{Const.} \exp(-\sin(x))}_{\text{Schon in } y_a \text{ drin!}}$$

Also: $\underline{\underline{y_s(x) = 1}}$!

4.

$y'' + 4y' + 4ay = 0$ Homogene DG 2. Ordnung mit konstanten Koeffizienten!

Ansatz: $y = e^{rx}$ $\Rightarrow$ $r^2 + 4r + 4a = 0$ (charakteristische Gleichung)

$r = -2 \pm \sqrt{4-4a}$

$= -2(1 \mp \sqrt{1-a})$

* $a < 1$ $\Rightarrow$ exponentiell wachsende / fallende Lösungen
* $a > 1$ $\Rightarrow$ keine reellen Nullstellen für r $\Rightarrow$ Schwingungen.

Für $a = 1$. $r = -2$

Also $y_1(x) = e^{-2x}$ ist eine Lösung.

Variation der Konstanten:

$y_2(x) = C(x)\, e^{-2x}$

$y_2'(x) = C' e^{-2x} - 2C e^{-2x}$

$y_2''(x) = C'' e^{-2x} - 4C' e^{-2x} + 4C e^{-2x}$

$$\Rightarrow \underbrace{C'' - \cancel{4C'} + 4C}_{y''\cdot e^{2x}} + \underbrace{\cancel{4C'} - 8C}_{4y'\cdot e^{2x}} + 4C = 0 \qquad C'' = 0 \qquad C = \underline{A\cdot x} + B$$

↑ nichts Neues

Die allgemeine Lösung: $y = C_1 y_1(x) + C_2 y_2(x) = \underline{\underline{C_1 e^{-2x} + C_2 x e^{-2x}}}$

5.

$M\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$ $\qquad M\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$M\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}M\left[\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}3-1\\4+0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$; $\quad M\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}M\left[\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}-1-3\\0-4\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$

$M_{11} = e_1^T M e_1 = (1\,0)\, M\begin{pmatrix}1\\0\end{pmatrix} = 1$

$M_{21} = e_2^T M e_1 = (0\,1)\, M\begin{pmatrix}1\\0\end{pmatrix} = 2$

$M_{12} = e_1^T M e_2 = (1\,0)\, M\begin{pmatrix}0\\1\end{pmatrix} = -2$

$M_{22} = e_2^T M e_2 = (0\,1)\, M\begin{pmatrix}0\\1\end{pmatrix} = -2$

$\Rightarrow M = \underline{\underline{\begin{pmatrix}1 & -2\\ 2 & -2\end{pmatrix}}}$ $\qquad \det M = -2+4 = 2 \neq 0$

$\Rightarrow$ ist regulär!

6.

$A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$; $B = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$; $\exp(\Theta M) = \sum_{k=0}^{\infty} \frac{1}{k!}\Theta^k M^k$

$$[A,B] = \underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}_{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}} - \underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} =: M$$

$$M^k = \begin{pmatrix}1^k & 0\\ 0 & (-1)^k\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & (-1)^k\end{pmatrix}$$

$$\exp(\Theta M) = \sum_{k=0}^{\infty}\frac{1}{k!}\Theta^k M^k = \begin{pmatrix}\sum_{k=0}^{\infty}\frac{1}{k!}\Theta^k & 0\\ 0 & \sum_{k=0}^{\infty}\frac{1}{k!}(-\Theta)^k\end{pmatrix} = \underline{\underline{\begin{pmatrix}e^{\Theta} & 0\\ 0 & e^{-\Theta}\end{pmatrix}}}$$