EMTP I / 2. Klausur / 31.08.2010 / M.L.

1.

Maclaurin-Reihe: $f(x) = f(0) + f'(0)\cdot x + \frac{1}{2} f''(0)\, x^2 + \frac{1}{6} f^{(3)}(0)\, x^3 + \dots$ (1 Punkt)

$f(0) = \tanh(0) = 0$ (1 Punkt)

$f'(0) = \tanh'(0) = 1 - \tanh^2(0) = 1$ (1 Punkt)

$$\frac{d^2}{dx^2}\tanh x = \frac{d}{dx}\left\{1 - \tanh^2 x\right\} = -2\tanh(x)\,\tanh'(x)$$

$$= -2\tanh(x) + 2\tanh^3(x)$$

$\Rightarrow f''(0) = 0$ (1 Punkt)

$$\left.\frac{d^3}{dx^3}\tanh x\right|_{x=0} = -2\tanh'(0) + 6\,\underbrace{\tanh^2(0)}_{0}\,\tanh'(0) = -2$$

$\Rightarrow \tanh x = x - \frac{1}{3}x^3 + \dots$

2.

$[p^2+q^2+2pq\cos\theta]_{min} = p^2+q^2-2pq = (p-q)^2 > 0$. $[p^2+q^2+2pq\cos\theta]_{max} = p^2+q^2+2pq = (p+q)^2 > 0$

keine Probleme solange $p \neq q$ gilt!

$$I_a = \int_0^\pi \frac{d\theta\,\sin\theta}{p^2+q^2+2pq\cos\theta} \quad ; \quad z = \cos\theta, \quad dz = -\sin\theta\, d\theta \qquad \theta = 0 \Rightarrow z = 1, \quad \theta = \pi \Rightarrow z = -1$$

$$= \int_{-1}^{+1} \frac{dz}{p^2+q^2+2pqz} = \frac{1}{2pq}\Big[\ln|p^2+q^2+2pqz|\Big]_{-1}^{+1} = \frac{1}{2pq}\ln\left|\frac{(p+q)^2}{(p-q)^2}\right| = \frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|$$

(2 Punkte)

$$I_b = \int_0^\pi \frac{d\theta\,\sin\theta}{(p^2+q^2+2pq\cos\theta)^2} = \int_{-1}^{+1}\frac{dz}{(p^2+q^2+2pqz)^2} = \frac{1}{2pq}\left[-\frac{1}{p^2+q^2+2pqz}\right]_{-1}^{+1}$$

$$= \frac{1}{2pq}\left[\frac{1}{(p-q)^2} - \frac{1}{(p+q)^2}\right] = \frac{1}{2pq}\cdot\frac{\cancel{p^2}+2pq+\cancel{q^2}-\cancel{p^2}+2pq-\cancel{q^2}}{(p^2-q^2)^2}$$

(2 Punkte)

$$= \frac{2}{(p^2-q^2)^2}$$

3.

$$F(y,x) = \int_0^y dt\, \frac{\sin(xt)}{t}$$

(1½ Punkte)

$$\partial_y F = \frac{\partial F}{\partial y} = \frac{\sin(xy)}{y}$$

$$\partial_x F = \frac{\partial F}{\partial x} = \int_0^y \frac{dt}{\cancel{t}}\cdot\cos(xt)\cdot\cancel{t} \overset{\substack{s = xt\\ ds = x\,dt}}{=} \frac{1}{x}\int_0^{xy} ds\,\cos(s) = \frac{1}{x}\Big[\sin(s)\Big]_0^{xy} = \frac{\sin xy}{x}$$

(1½ Punkte)

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial_x F}{\partial_y F} = -\frac{\frac{\sin xy}{x}}{\frac{\sin xy}{y}} = -\frac{y}{x}$$

(1 Punkt)

4.

$y_{n+2} + y_{n+1} - 6y_n = 0 \quad , n \geq 0$

Eine lineare DG $\Rightarrow$ y_n Lösung $\Rightarrow$ $C \cdot y_n$ Lösung (1 Punkt)

Ansatz: $y_n = r^n$

$\Rightarrow \quad r^{n+2} + r^{n+1} - 6r^n = 0 \quad , n \geq 0$

Um eine nichttriviale Lösung zu finden, nehme $r \neq 0$ an $\Rightarrow r^n \neq 0 \Rightarrow$ dividiere durch r^n

$\Rightarrow \quad r^2 + r - 6 = 0 \qquad r = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+6} = -\frac{1}{2} \pm \frac{5}{2}$ (2 Punkte)

$r = 2 \quad \vee \quad r = -3$

$\Rightarrow \quad y_n = \underline{C_1 \cdot 2^n + C_2 \cdot (-3)^n} \quad ; \; C_1, C_2 = \text{const.}$ (1 Punkt)

5.

$P(\vec{e}_x, \vec{e}_y, \vec{e}_z) = (\vec{e}_y, \vec{e}_x, \vec{e}_z)$

$P_{ij} = \vec{e}_i \cdot P\vec{e}_j \qquad \Rightarrow \qquad \begin{aligned} P_{i1} &= \vec{e}_i \cdot P\vec{e}_x = \vec{e}_i \cdot \vec{e}_y = \delta_{i2} \\ P_{i2} &= \vec{e}_i \cdot P\vec{e}_y = \vec{e}_i \cdot \vec{e}_x = \delta_{i1} \\ P_{i3} &= \vec{e}_i \cdot P\vec{e}_z = \vec{e}_i \cdot \vec{e}_z = \delta_{i3} \end{aligned} \qquad \Rightarrow P_{ij} = \begin{array}{c} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$

$i \downarrow \xrightarrow{j}$

(2 Punkte)

(1 Punkt)

$\text{Det } P_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \underline{\underline{-1}}$

$P \cdot P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (P^{-1})_{ij} = P_{ij}$ (1 Punkt)

6.

$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \det(A - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} = -\lambda^3 + 1 = 0$

$\lambda^3 = 1 \qquad \lambda = +1 \qquad \left(\text{oder } e^{\pm \frac{i2\pi}{3}} \in \mathbb{C} \right)$ (1 Punkt)

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \begin{cases} c = a \\ a = b \\ b = c \end{cases} \qquad \Rightarrow \quad v = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (1 Punkt)

(1 Punkt)

$B = \begin{pmatrix} 2 & +4 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \det(B - \lambda \mathbb{1}) = \begin{vmatrix} 2-\lambda & +4 & 0 \\ -1 & -3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\left[(\lambda-2)(\lambda+3)+4\right] = 0$

$\lambda = 5 \quad \vee \quad \lambda^2 + \lambda - 2 = 0 \qquad \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = -\frac{1}{2} \pm \frac{3}{2} = \begin{cases} +1 \\ -2 \end{cases}$

$\lambda = 5: \quad \begin{pmatrix} 2 & 4 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 5 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \begin{cases} 2a + 4b = 5a \\ -a - 3b = 5b \\ 5c = 5c \end{cases} \quad \begin{aligned} 4b &= 3a \\ 8b &= -a \\ 5c &= 5c \end{aligned} \quad \Rightarrow a = b = 0 \quad \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = +1: \quad \begin{pmatrix} 2 & 4 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \begin{cases} 2a + 4b = a \\ -a - 3b = b \\ 5c = c \end{cases} \quad \Rightarrow c = 0 \quad \begin{aligned} 4b &= -a \\ -a &= 4c \end{aligned} \quad \Rightarrow \frac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -2: \quad \begin{pmatrix} 2 & 4 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \begin{cases} 2a + 4b = -2a \\ -a - 3b = -2b \\ 5c = -2c \end{cases} \quad \Rightarrow c = 0 \quad \begin{aligned} 4b &= -4a \\ -a &= b \end{aligned} \quad \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(2 Punkte)

Eigenvektoren nicht orthogonal, weil B nicht symmetrisch. (1 Punkt)