EMTP I / 1. Klausur / 03.03.2010 / M.L.

(1.)

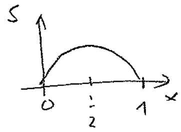
$S(x) = -x\ln x - (1-x)\ln(1-x) \quad , \; 0 \le x \le 1$

$S'(x) = -\ln x - 1 + \ln(1-x) + 1 = \ln\frac{1-x}{x}$ (1 Punkt)

$S'(x) = 0 \iff \ln\frac{1-x}{x} = 0 \iff \frac{1-x}{x} = e^0 = 1 \qquad 1-x = x \qquad \underline{\underline{x = \frac{1}{2}}}$ (1 Punkte)

$S(\frac{1}{2}) = -\frac{1}{2}\ln(\frac{1}{2}) - \frac{1}{2}\ln(\frac{1}{2}) = -\ln(\frac{1}{2}) = \ln 2 > 0$

$S(0) = S(1) = 0$

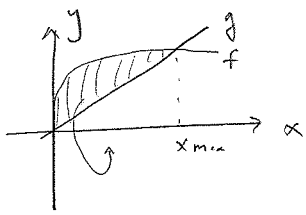


$x = \frac{1}{2}$ ist <u>Maximum</u>

$x = 0,1$ sind <u>Minimal</u>

(2 Punkte).

(2)



(1 Punkt)

$\sqrt{x_{max}} = x_{max} \iff 1 = \sqrt{x_{max}} \Rightarrow x_{max} = 1$ (1 Punkt)

$V = \int_0^1 dx \; \pi \left\{ [f(x)]^2 - [g(x)]^2 \right\}$ ($\frac{1}{2}$ Punkt)

$= \int_0^1 dx \; \pi \left\{ x - x^2 \right\} = \pi \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{\pi}{6}$

($\frac{3}{2}$ Punkte)

(3)

$\frac{dT}{dt} = -k(T - T_0) \qquad y = T - T_0 \qquad \frac{dy}{dt} = \frac{dT}{dt}$

$\Rightarrow \frac{dy}{dt} + ky = 0$ lineare DG 1. Ordnung! (1 Punkt)

$y = e^{rt} \qquad (r+k)e^{rt} = 0 \qquad r = -k$

$\Rightarrow$ allgemeine Lösung ist $y_a(x) = Ce^{-kt}$ ($1\frac{1}{2}$ Punkte)

$t = 0\,\text{min} \Rightarrow 300\,K - 270\,K = C \iff C = 30\,K$ ($\frac{1}{2}$ Punkt)

$t = 3\,\text{min} \Rightarrow 280\,K - 270\,K = C \cdot e^{-k\cdot(3\,\text{min})}$

$e^{-k(3\,\text{min})} = \frac{10\,K}{30\,K} = \frac{1}{3}$

$k(3\,\text{min}) = -\ln\frac{1}{3} = \ln 3$

$k = \frac{\ln 3}{3\,\text{min}}$ (1 Punkt)

④ $x^2 y'' + xy' - y = x \qquad u = \ln x \qquad x = e^u$

$$y' = \frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du} = \frac{1}{x}\frac{dy}{du}$$

$$y'' = \frac{d}{dx}\left\{\frac{1}{x}\frac{dy}{du}\right\} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{du}{dx}\frac{d^2y}{du^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}$$

$$\Rightarrow \frac{d^2y}{du^2} - \cancel{\frac{dy}{du}} + \cancel{\frac{dy}{du}} - y = 0$$ lineare DG 2. Ordnung! (2 Punkte)

Allgemeine Lösung der homogenen Gleichung:

$$y = Ae^u + Be^{-u} = \underline{\underline{Ax + \frac{B}{x}}}$$ (2 Punkte)

$\uparrow$ 1 Pkt $\qquad \uparrow$ 1 Pkt

Spezielle Lösung: $y = ue^u$

$y' = ue^u + e^u$

$y'' = ue^u + 2e^u$

$\Rightarrow y_s(u) = \frac{1}{2}ue^u \quad ; \quad \underline{\underline{y_s(x) = \frac{1}{2}x\ln x}}$ (1 Punkt)

⑤ $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ (1 Punkt)

$$\det A \underset{\substack{\uparrow \\ \text{Laplace,} \\ \text{1. Spalte}}}{=} \underbrace{0\cdot\begin{vmatrix} 0 & c \\ -c & 0\end{vmatrix}}_{0} + \underbrace{a\begin{vmatrix} a & b \\ -c & 0\end{vmatrix}}_{abc} \underbrace{-b\begin{vmatrix} a & b \\ 0 & c\end{vmatrix}}_{-bac} = 0$$ (2 Punkte)

$\det\begin{pmatrix} 0 & a \\ -a & 0\end{pmatrix} = a^2 \neq 0$ (1 Punkt)

⑥ $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\qquad \det(A - \lambda \mathbb{1}) = (1-\lambda)^2 - 4 = 0$

$\Leftrightarrow \lambda^2 - 2\lambda - 3 = 0 \quad \lambda = 1 \pm \sqrt{1+3} = \begin{cases} 3 \\ -1 \end{cases}$ (1 Punkt)

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1\begin{pmatrix} a \\ b \end{pmatrix} \qquad a + 2b = -a \qquad a = -b \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix} \qquad a + 2b = 3a \qquad a = b \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (1 Punkt)

Sei $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$; $P^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = P^{-1}$

Dann ist $A \cdot P = \left(-1v_1, 3v_2\right) = (v_1\ v_2)\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = P\Lambda$

$\Leftrightarrow A = P \Lambda P^{-1}$ (2 Punkte)

$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \quad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$